CDI-II — Prática F7 21/4/21

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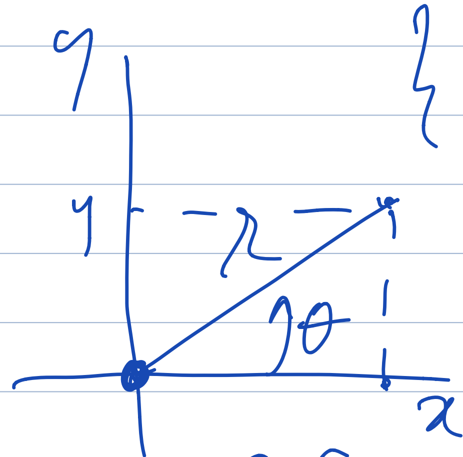
Ficha 7



$$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

$(x,y) = g(r,\theta)$

$$\iint_X f(x,y)\,dx\,dy = \iint_T f(g(r,\theta))\underbrace{|\det Dg(r,\theta)|}_{r}\,dr\,d\theta$$

$X = g(T)$

$$dx\,dy \longleftrightarrow r\,dr\,d\theta$$

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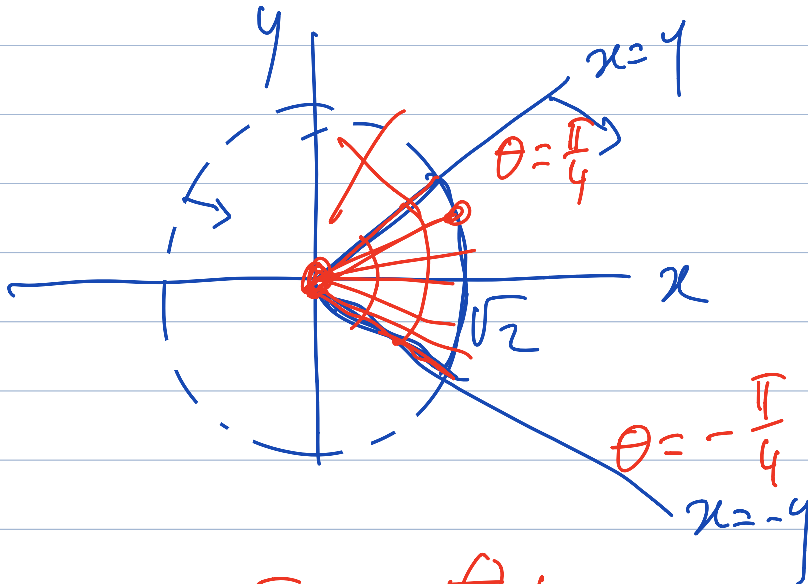
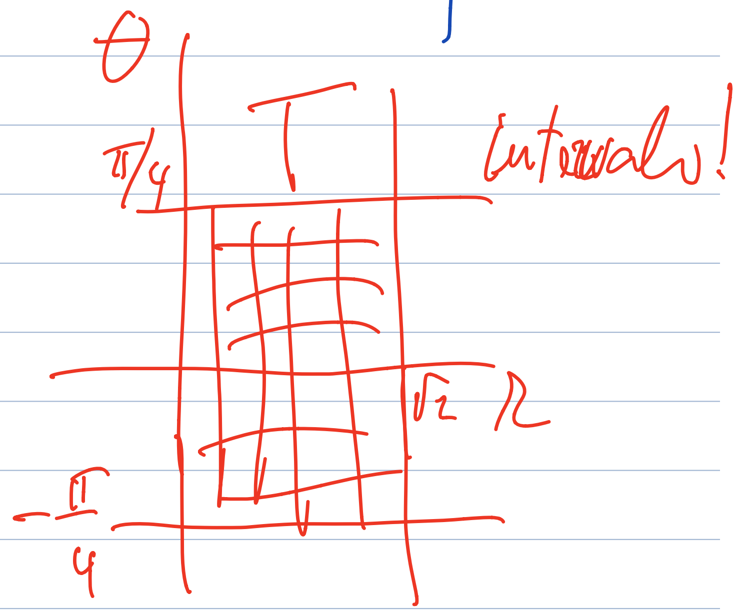
1-a) $x^2+y^2<2$; $x>|y|$



$$\begin{cases} -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ 0 < r < \sqrt{2} \end{cases}$$



$$\iint_X f(x,y)\,dx\,dy = \iint_T f(g(r,\theta))\, r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\int_0^{\sqrt{2}} f(g(r,\theta))\, r\, dr \right) d\theta$$

ou

$$= \int_0^{\sqrt{2}} \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(r\cos\theta, r\sin\theta)\, r\, d\theta \right) dr$$

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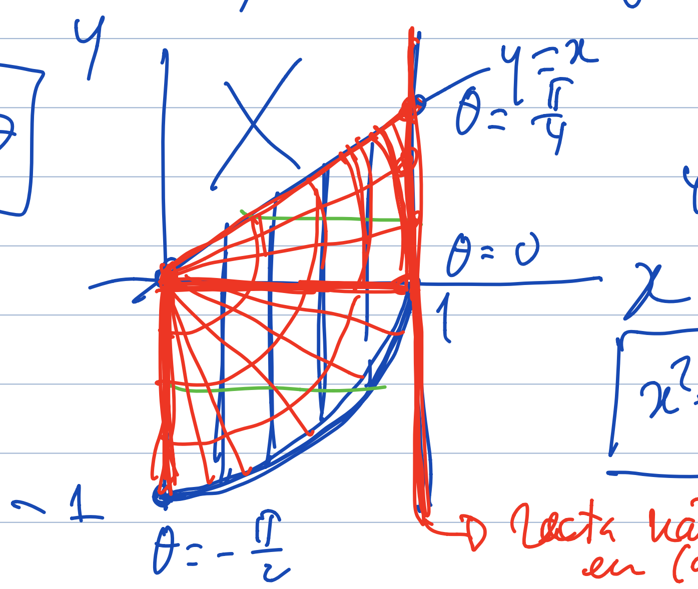
1-c) $X: 0 < x < 1 \ ; \ -\sqrt{1-x^2} < y < x$

$\boxed{x = 1 = r\cos\theta}$

$\boxed{r = \frac{1}{\cos\theta}}$



$y = -\sqrt{1-x^2}$

$y^2 = 1 - x^2, y < 0$

$\boxed{x^2 + y^2 = 1, y < 0}$

$$\int_{-\frac{\pi}{2}}^{0}\left(\int_{0}^{1} f(r\cos\theta, r\,\mathrm{sen}\,\theta)\, r\, dr\right) d\theta \ +$$

$$+ \int_{0}^{\frac{\pi}{4}}\left(\int_{0}^{\frac{1}{\cos\theta}} f(r\cos\theta, r\,\mathrm{sen}\,\theta)\, r\, dr\right) d\theta$$

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$$\int_{-\frac{\pi}{2}}^{0}\left(\int_{0}^{1} f(\ldots)\, r\, dr\right) d\theta \ +$$

$$+ \int_{0}^{1}\left(\int_{0}^{x} f(x,y)\, dy\right) dx$$

$$\int_0^1 \left(\int_{\boxed{-\sqrt{1-x^2}}}^{x} f(x,y)\,dy \right) dx$$

dificuldade.

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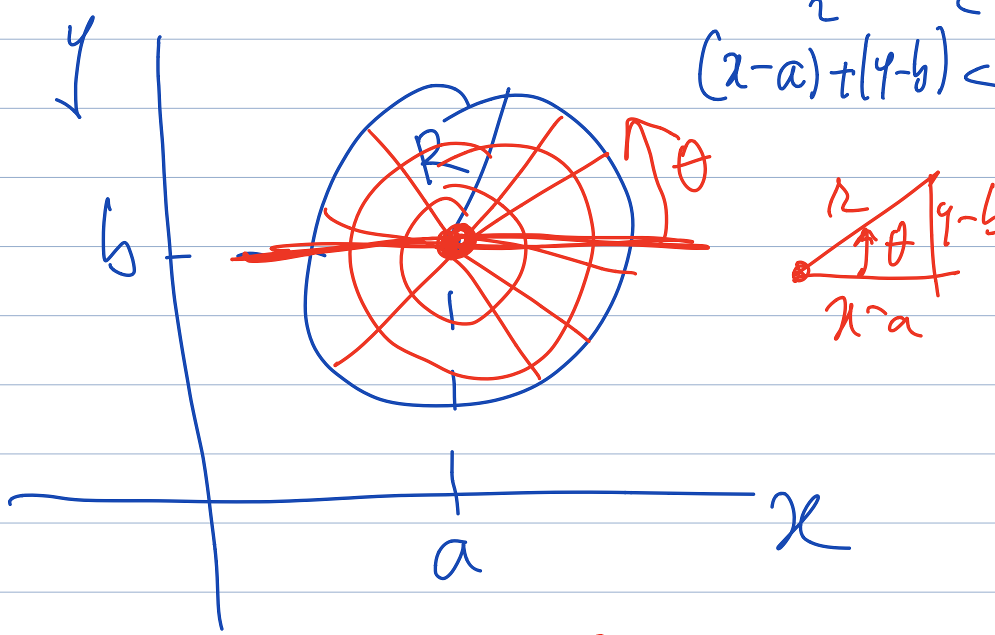
$$(x-a)^2+(y-b)^2<R^2$$

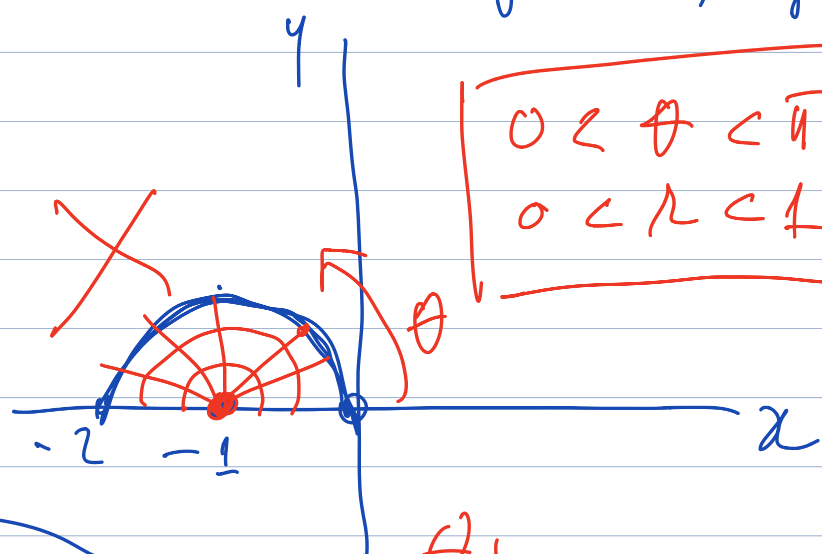


$$\begin{cases} x-a = r\cos\theta \\ y-b = r\,\text{sen}\,\theta \end{cases} \qquad \begin{cases} x = a + r\cos\theta \\ y = b + r\,\text{sen}\,\theta \end{cases}$$

$$(x,y) = g(r,\theta) = (a + r\cos\theta,\ b + r\sin\theta)$$

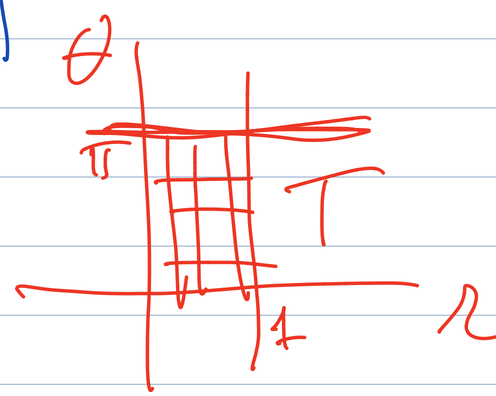
$$\boxed{\det Dg(r,\theta) = r}$$

2-c) $(x+1)^2 + y^2 < 1\ ;\ y > 0$



$$\boxed{\begin{array}{l} 0 < \theta < \pi \\ 0 < r < 1 \end{array}}\ T$$

$$\begin{array}{l} a = -1 \\ b = 0 \end{array}$$



$$\iint_X \underbrace{(x^2+y^2-1)}_{f(x,y)}\, dx\, dy =$$

$$= \iint_T f(g(r,\theta))\, r\, dr\, d\theta$$

$$\begin{cases} x = -1 + r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

$$= \int_0^{\pi} \left(\int_0^1 \left((r\cos\theta - 1)^2 + r^2\,\mathrm{sen}^2\theta - 1 \right) r\, dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^1 \left(r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\,\mathrm{sen}^2\theta - 1 \right) r\, dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^1 \left(r^2 - 2r\cos\theta \right) r\,dr \right) d\theta$$

etc...

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2-d)

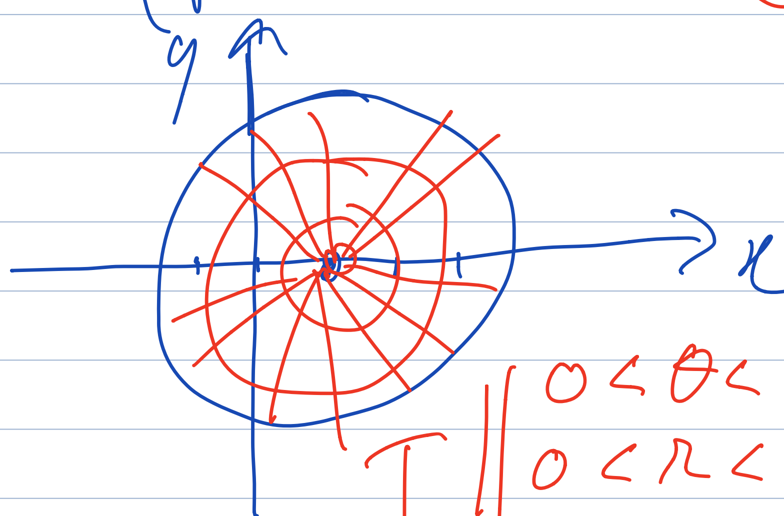
$$X: \quad (x-1)^2 + y^2 < \frac{\pi^2}{4}$$

$a = 1$

$b = 0$

$$\begin{cases} x = 1 + r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{cases}$$

r

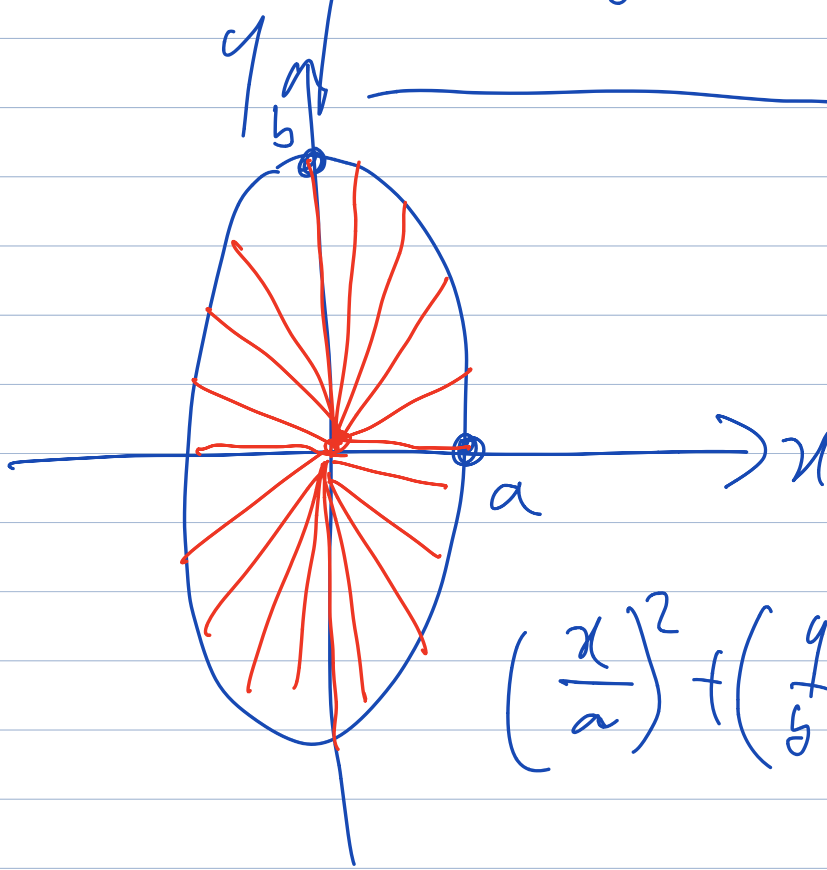


$$T \left\| \begin{array}{l} 0 < \theta < 2\pi \\ 0 < r < \frac{\pi}{2} \end{array} \right.$$

$$\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}} \operatorname{sen}(r^2)\, r\, dr\right) d\theta \quad \text{etc} \ldots$$

— u —

Elipse: $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\leq 1}$



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\leq 1$$

$$\begin{cases} \dfrac{x}{a} = r\cos\theta \\ \dfrac{y}{b} = r\,\mathrm{sen}\,\theta \end{cases}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r^2 \leq 1$$

$$\boxed{0 < r < 1}$$

$$\begin{cases} x = ar\cos\theta \\ y = br\,\mathrm{sen}\,\theta \end{cases}$$

$$\frac{y}{x} = \frac{b}{a}\tan\theta \iff \frac{ay}{bx} = \tan\theta$$

$$\boxed{\theta = \arctan\left(\frac{ay}{bx}\right)}$$

$(r, \theta) \equiv$ coordenadas polares da elipse.

$$g(r,\theta) = (x,y) = (a r\cos\theta,\ b r \operatorname{sen}\theta)$$

$$\boxed{\det Dg(r,\theta) = abr}$$ exercício.

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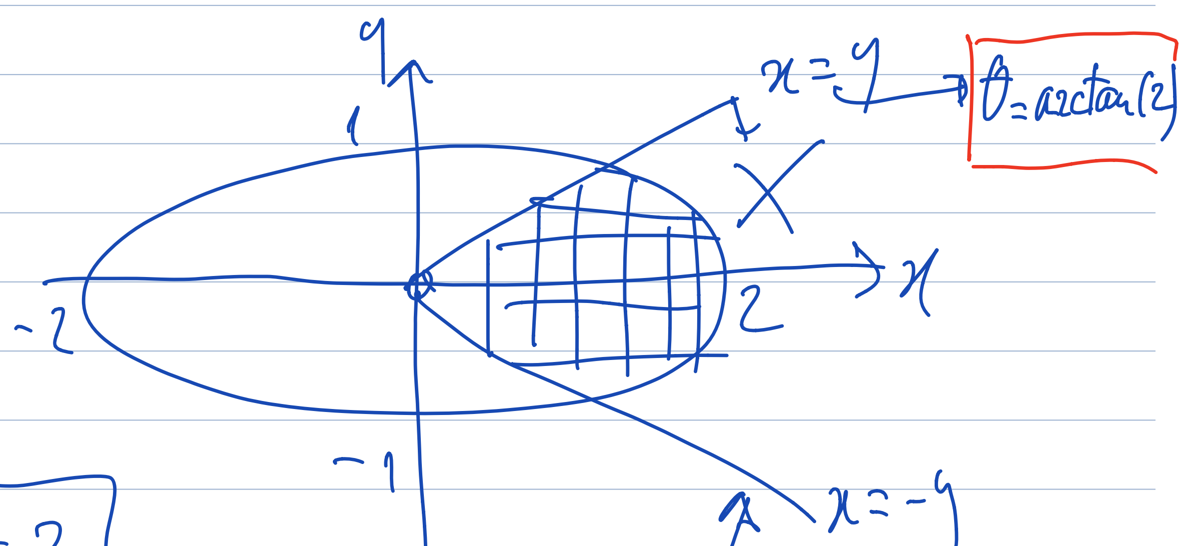
2-e) $\dfrac{x^2}{4} + y^2 \le 1$; $x > |y|$



$$\boxed{\begin{array}{l} a = 2 \\ b = 1 \end{array}}$$

$$\theta = \arctan\left(\frac{2y}{x}\right) =$$

$$-\arctan(2) = \arctan(-2)$$

$$\left| \begin{array}{l} -\arctan 2 < \theta < \arctan 2 \\ 0 < r < 1 \end{array} \right. \quad \boxed{2r}$$

$$\iint_X 1 = \int_X 1 = \mathrm{vol}_2(X)$$

$$= \int_{-\arctan 2}^{\arctan 2} \left(\int_0^1 2r\,dr \right) d\theta$$

$$= 2\arctan(2).$$

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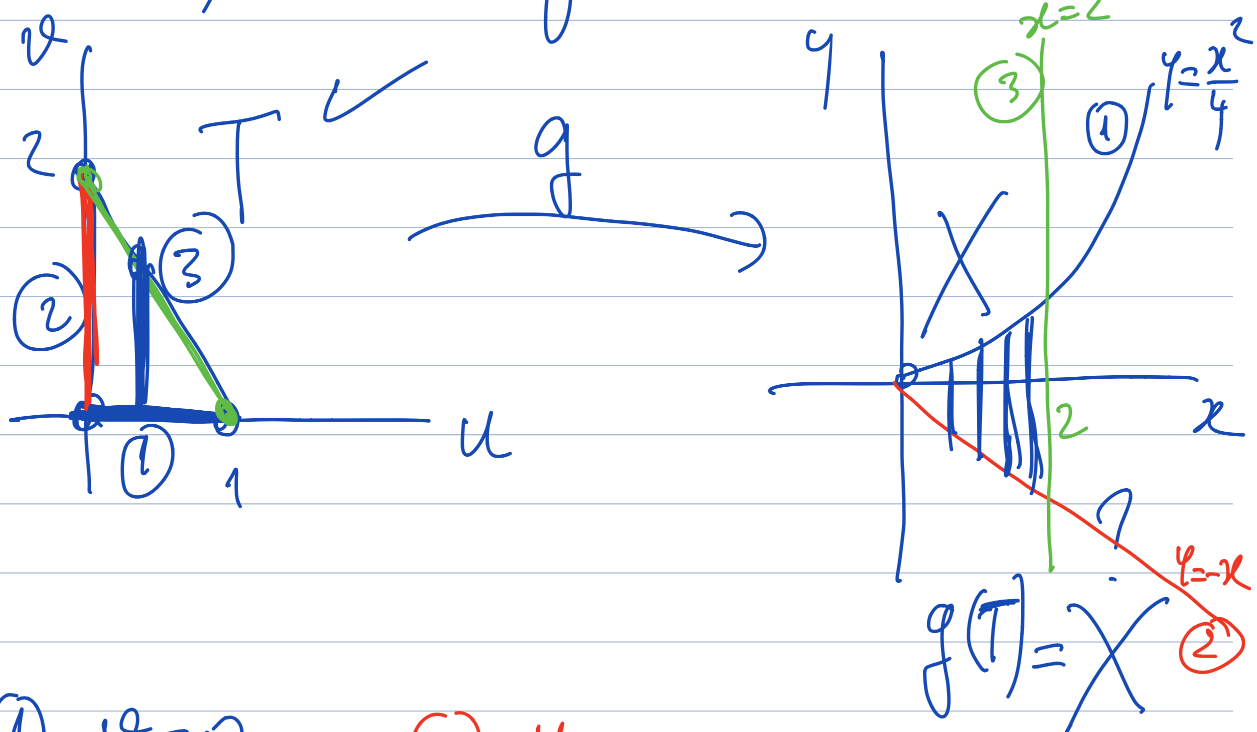
$$\mathbb{R}^n: \int_X f = \iint_X \cdots \int f(x_1, \dots, x_n)\,dx_1 \cdots dx_n$$

$$3\text{-}a)\quad (x,y) = g(u,v) = (\boxed{2u+v},\ u^2-v)$$



① $v=0$

$$\downarrow$$

$$\begin{cases} x = 2u \\ y = u^2 \end{cases}$$

$$\begin{cases} u = \frac{x}{2} \\ \boxed{y = \frac{x^2}{4}} \end{cases}$$

② $u=0$

$$\downarrow$$

$$\begin{cases} x = v \\ y = -v \end{cases}$$

$$\downarrow$$

$$\boxed{y = -x}$$

③ $2u+v=2$

$$\downarrow$$

$$\boxed{x = 2}$$

3-b) $\iint_X f(x,y)\,dx\,dy$

or $\int_X f$

$$\boxed{X = g(T)}$$

$$\iint_X f(x,y)\,dx\,dy = \iint_T f\big(g(u,v)\big)\,\big|\det Dg(u,v)\big|\,du\,dv$$

$$Dg(u,v) = \begin{bmatrix} 2 & 1 \\ 2u & -1 \end{bmatrix}$$

$$|\det Dg(u,v)| = |-2-2u| = 2+2u$$

$$\iint_X \frac{1}{\sqrt{x+y+1}}\,dx\,dy = \iint_T \frac{1}{\sqrt{2u+v+u^2-v+1}}(2+2u)\,du\,dv$$

$$= \int_0^1 \left(\int_0^{2-2u} \frac{2(1+u)}{\underbrace{\sqrt{u^2+2u+1}}_{\sqrt{(u+1)^2}}}\,dv \right) du$$

$$= \int_0^1 \left(\int_0^{2-2u} 2\,dv \right) du$$

... etc

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$$\int_X f \equiv \iint_X f \equiv \iint_X f(x,y)\,dx\,dy$$

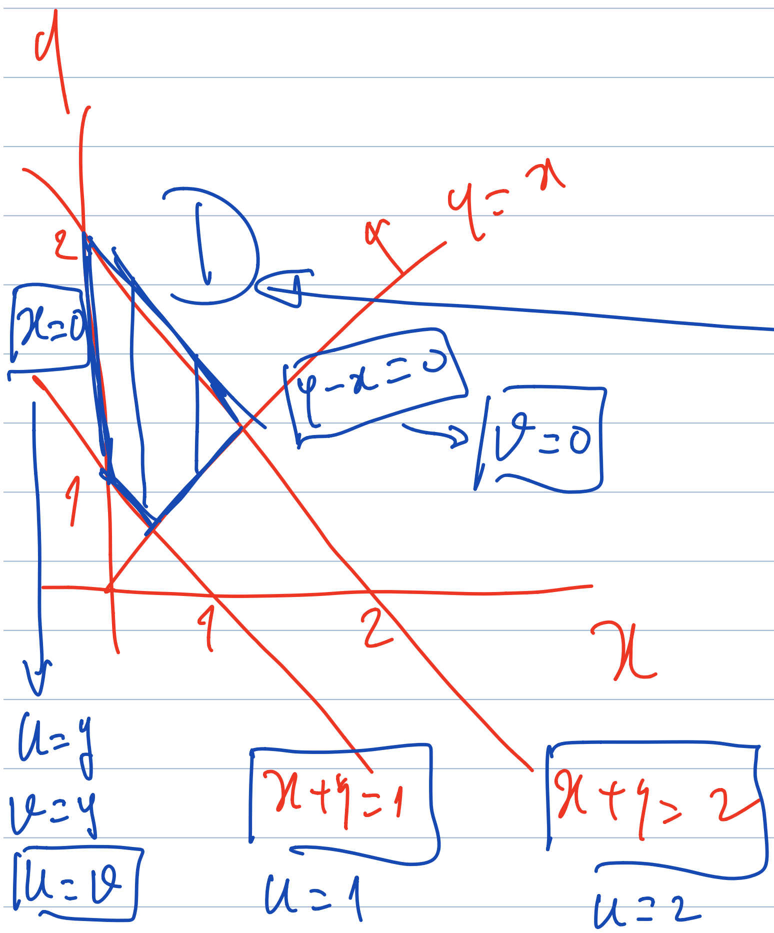
Mudar de variáveis serve para "simplificar" X e/ou f.

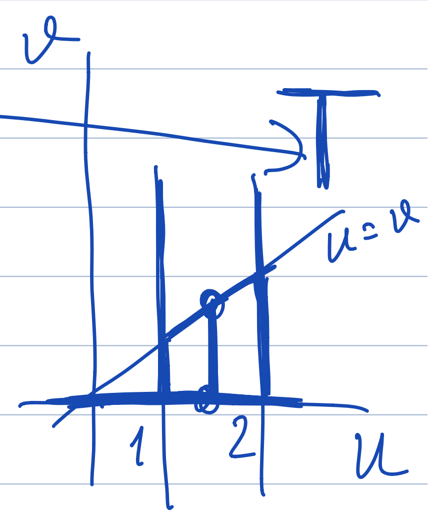
X: $\quad 1 < \underbrace{\boxed{x+y}}_{u} < 2 \quad ; \quad 0 < x < y$

f: $\quad f(x,y) = \underbrace{(y^2 - x^2)}_{\underbrace{\boxed{(y-x)}}_{v}\underbrace{(y+x)}_{u}} \cos(\underbrace{x+y}_{u})^4$

$$\begin{cases} u = x+y \\ v = y-x \end{cases} \qquad \det\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \neq 0$$

$\frac{1}{2}$ "preço a pagar para mudar de (x,y) para (u,v)".





$$f(g(u,v)) = uv\cos(u^4)$$

"simples"

$$T = \text{"simples"}$$

$$\int_D f = \frac{1}{2}\iint_T uv\cos(u^4)\,du\,dv$$

$$= \frac{1}{2}\int_1^2\left(\int_0^u uv\cos(u^4)\,dv\right)du$$

... etc